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Journal of Sound and Vibration (1972) **20** (3), 407–412

ON VIBRATING STRINGS AND INFORMATION THEORY

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(*Received 31 July 1971, and in revised form 30 November 1971*)

The problem addressed *in concreto* is the relation of information provided by vibrating strings to that provided by systems describable with equations of one degree of freedom. Whereas the mathematical physics of vibrating strings is based on the wave equation—a second-order differential equation of at least two degrees of freedom—a quantum model of information theory has only been considered for a mechanical system of one degree of freedom.

The solution obtained *in abstracto* is: a complete signal representation information-wise exists in Hilbert space. With an increasing degree of freedom to vary of any system corresponds an increasing number of phase representations of the signal producible by the system in that space.

A corollary conclusion is: the spectral analysis of signals based on Wiener's Fourier method is incomplete. A complete conception of information based on complex signals in two subspaces of Hilbert space is more general. A spectral analysis of the information available in signals in Hilbert space is thus possible giving a possible explanation for the timbre of a sound.

INTRODUCTION

Lord Rayleigh's two volumes on the theory of sound [1] treat the stimuli arising from vibrating strings in detail. Since such stimuli arise from simple mechanical systems, their treatment in an information theory analysis should also provide a simple description. The physics of vibrating strings is based on a wave equation of two degrees of freedom and the question arises: how can the information produced by a system describable by a wave equation be compared and contrasted with that produced by a system describable, e.g., by an equation of one degree—or *n*-degrees—of freedom? This paper presents a general solution to the problem.

First, however, let us distinguish our approach to information theory from that of Shannon. Shannon's work [2] involves the resolution of uncertainties concerning final outcome in the face of a repertoire of possible occurrences, these occurrences possibly varying in their probability of occurrence. The accent is on hypothetical transmission of events defined abstractly, so that a temporal aspect is given to the theory. Thus, his work may more properly be termed transmission theory—to distinguish it from information theory based strictly on energy distribution—a distinction he was aware of. This paper will not be concerned with this latter type of information theory but rather will address the analog information theory of Gabor [3].

ARGUMENT

We shall commence by defining a signal completely in Hilbert space. The complex numbers α and β are defined so that β is the complex conjugate of α:

$$\begin{aligned} \alpha &= \Delta f.\Delta t + jf_0.t_0, \\ \beta &= \Delta f.\Delta t - jf_0.t_0, \end{aligned} \qquad (1)$$

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where Δf, signal bandwidth, and Δt, signal duration, are reciprocally related for a minimum value by the uncertainty condition,

$$\Delta f . \Delta t = 1/2. \tag{2}$$

[3, 4] and f_0, the average signal frequency, and t_0, the average period, are related by the logical relation, $f_0 . t_0 = 1/2$ for a minimum value. A signal is the inner product $\alpha\alpha^*$ in subspaces M_1, M_2—spaces square summable and defined by

$$\mathsf{H} = \mathsf{M}_1 . \mathsf{M}_2. \tag{3}$$

Space H is a Hilbert space. Such a signal is a bilinear functional on the spaces M_1 and M_2 and, by the uncertainty condition stated above, is *symmetric*.

Now, the signal bandwidth $\times$ duration uncertainty product defined as the minimum information quantum or *logon*—$\Delta f . \Delta t$ = one logon at the minimum value of 1/2 [3]—can, for example, be used to represent the output of a system described by a differential equation of the form [5]

$$A\frac{d^2 x}{dt^2} + D\frac{dx}{dt} + Kx = e^{-\alpha}, \tag{4}$$

where $f_0 = (1/2\pi)\sqrt{K/A}$, $\Delta f . \Delta t = (\alpha/2\pi)\sqrt{(e^{\alpha} - 1)}$, and $e^{-\alpha}$ expresses the change in energy discernible by the system.

I would like to emphasize that the expression for $\Delta f . \Delta t$ makes no reference to the system parameters but only to the energy resolution parameters: e^{α} and α. Clearly, these parameters may be related in any way to any system considered, in which case this simple expression for $\Delta f . \Delta t$ used in equation (4) will represent summed effects and hide the unique relation of the energy resolution parameters to parameters of the system considered.

A system such as that of equation (4) has but one degree of freedom, and from the above consideration an information analysis of the signals producible by it can be represented as a set of numbers in Hilbert space. We shall now turn to vibrating strings which are systems of more than one degree of freedom and for which a more complex analysis is needed.

By a theorem of E. Schmidt [6, p. 243], every function, say $A(x, y)$, which is square summable and symmetric can be developed, in the sense of convergence in the mean, into the series

$$A(x, y) = \sum_i \mu_i \phi(x)_i \overline{\phi_i(y)}, \tag{5}$$

where $\phi_i(x)$ denotes the orthonormal sequence of characteristic functions and μ_i the sequence of corresponding characteristic values of transformation A generated by the kernel $A(x, y)$. In the case of a vibrating string, the plane (x, y) is the plane of vibration; the string is assumed fixed at the points (0,0) and (1,0) and the string's movements described by a function, $y(x, t)$. The specific analysis is as follows.

The second-order differential equation describing the behavior of a vibrating string is the wave equation,

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}, \tag{6}$$

a solution for which is

$$y(x, t) = \sum_n (a_n \cos p_n t + b_n \sin p_n t) u_n(x). \tag{7}$$

We shall show that the expressions $\sum_n (a_n \cos p_n t + b_n \sin p_n t)$ are characteristic functions $c_n(t) = \overline{u_n(x)}$.

From the consideration that the function $y(x,t)$ corresponds to the movements of which the string is capable in the time interval $0 > t > T$, it may be defined with respect to its ability to make the following integral stationary [6]:

$$\int_0^T (E_c - E_p)\,\mathrm{d}t = \int_0^T \mathrm{d}t\left[\frac{1}{2}\int_0^1\left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}m(x) - \frac{1}{2}\int_0^1\left(\frac{\partial y}{\partial x}\right)^2 c^2\,\mathrm{d}m(x)\right], \tag{8}$$

where

$$E_c = \frac{1}{2}\int_0^1\left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}m(x) \tag{9}$$

is the kinetic energy of the string,

$$E_p = \frac{1}{2}\int_0^1\left(\frac{\partial y}{\partial x}\right)^2 c^2\,\mathrm{d}m(x) \tag{10}$$

is the potential energy of the string and $\mathrm{d}m(x)$ is the mass borne by the segment $(0,x)$ of the string. A function $\eta(x,t)$ is defined satisfying the limit conditions: $\eta(x,0)=0$; $\eta(x,T)=0$.

Therefore, equation (8) is redefined:

$$\int_0^T \mathrm{d}t\left[\int_0^1 \frac{\partial y}{\partial t}\frac{\partial \eta}{\partial t}\,\mathrm{d}m(x) - \int_0^1 \frac{\partial y}{\partial x}\frac{\partial \eta}{\partial x}\,\mathrm{d}m(x)\right] = 0 \tag{11}$$

and by equation (5), $y(x,t)$ may be defined in the metric of Hermitian (and Hilbert) spaces D and H, if x and t are symmetric:

$$y(x,t) = \sum c_k(t)\,u_k(x), \tag{12}$$

$$\frac{\partial y}{\partial t}(x,t) = \sum d_k(t)\,\phi_k(x). \tag{13}$$

It can be shown that the coefficients

$$c_k(t) = (y, u_k)_D \tag{14}$$

and

$$d_k = \left(\frac{\partial y}{\partial t}, \phi_k\right)_H \tag{15}$$

are related:

$$d_k(t) = \frac{1}{p_k}\frac{\partial c_k(t)}{\partial t}. \tag{16}$$

By redefining the function $\eta(x,t)$ satisfying the limit conditions,

$$\eta(x,t) = \gamma_n(t)\,u_n(x), \tag{17}$$

where $\gamma_n(t)$ is a function which is zero at points $t=0$ and $t=T$, and possesses a continuous derivative, but is otherwise quite arbitrary, equation (11) may be written as

$$\int_0^T \mathrm{d}t\left(\frac{\partial c_n}{\partial t}\frac{\partial \gamma_n}{\partial t}\Big/ p_n^2 - c_n\gamma_n\right) = 0, \tag{18}$$

where

$$c_n(t) = a_n\cos p_n t + b_n \sin p_n t. \tag{19}$$

These functions $c_n(t)$ are characteristic functions and have a representation in the metric of the Hilbert space D. The characteristic functions, $u_k(x)$, of equation (12) are an orthonormal sequence in the space H and if the string is homogeneous they satisfy the differential equation

$$\frac{\partial^2 u}{\partial t^2} + p^2 u = 0. \tag{20}$$

We shall now turn to information representation in general.

Any system gives rise to an energy distribution of information in which the triple product of resolution limits, U, is defined in the form

$$U = \frac{\Delta E}{E_0} \cdot \frac{\Delta f}{f_0} \cdot \frac{\Delta t}{T_0}, \tag{21}$$

where ΔE is the least energy change resolved by the system, Δf is the least frequency change resolved by the system, Δt is the least time interval during which the system is capable of changing its energy storage, E_0 is the initial amount of energy stored in the analyzer, f_0 is the natural frequency of the analyzer and T_0 is the natural period of the system.

Such a triple product is the volume of a region within which no change of state may be observed. U is called the indication limit [7]. The uncertainty relation is

$$\Delta f . \Delta t = \frac{1}{\pi}\left(\frac{\Delta E}{E_0}\right)^{1/2} \cdot \left(\frac{1}{1 - \Delta E/E_0}\right)^{1/2} \ln\left(\frac{1}{1 - \Delta E/E_0}\right)^{1/2}. \tag{22}$$

The definition of information "quanta" in terms of equation (22) is based on the response of any system or analyzer which exhibits oscillatory behavior. We shall show that for vibrating strings it is a half truth.†

The system's information response defined by equation (22) has one degree of freedom as far as defining the number of logons is concerned: i.e., the right-hand side only is taken into consideration. The representation expressed in equation (12), on the other hand, is the solution to a wave equation and has two degrees of freedom. It is evident that whereas a description of the distribution of energy (by the $\Delta f . \Delta t$ relation) in a system of two degrees of freedom will also have two degrees of freedom to vary, a description of energy in a system of one degree of freedom will have but one degree of freedom to vary. As information theory is basically a description of *energy distribution* (although not of absolute energy amounts), equation (22) represents a summed result. Thus, corresponding to the characteristic functions u_k and c_k of equation (12) we have $(\Delta f . \Delta t)_u$ and $(\Delta f . \Delta t)_c$ [and equation (22) applies to $\sum_i (\Delta f . \Delta t)_i$]. Since $y(x, t) = \sum c_k(t) u_k(x)$ is symmetric, or (if the complex case is studied) Hermitian, it is evident that $(\Delta f . \Delta t)_u = \overline{(\Delta f . \Delta t)_c}$, and furthermore, $f_{0u} = \overline{f_{0c}}$ and $t_{0u} = \overline{t_{0c}}$. Thus, the preceding general information analysis in Hilbert space has two representations in the case of vibrating strings. This conclusion has interesting implications for the physics of hearing as the function of the cochlea is a hydrodynamic problem, whether the function describing the system is symmetric or not.

In what way, for example, will the cochlear response to (i) $y(t) = c_k(t)$, (ii) $y(x, t) = c_k(t) u_k(x)$ and (iii) $y(x, y, t) = c_k(t) u_k(x) v_k(y)$ differ? As $\Delta f . \Delta t = 1/2 = 1$ logon is based on energy resolution parameters alone [equations (4) and (22)], at first blush there would appear to be no difference. Yet the restriction that $\sum_i \Delta f_i . \Delta t_i$ cannot equal less than 1/2 (which is one logon) does not stipulate how the minimum energy resolution parameters may be displaced or related among parameters of the system.

Thus in case (i) we might have $(\Delta f . \Delta t)_c = \sum_i (\Delta f . \Delta t)_i = 1/2$ for a minimum value. In case (ii) we might have $(\Delta f . \Delta t)_c + (\Delta f . \Delta t)_u = \sum_i (\Delta f . \Delta t)_i = 1/2$, for a minimum value, where $(\Delta f . \Delta t)_c = \frac{3}{10}$ and $(\Delta f . \Delta t)_u = \frac{2}{10}$; or, $(\Delta f . \Delta t)_c = \frac{2}{10}$ and $(\Delta f . \Delta t)_u = \frac{3}{10}$; or any other

† The arbitrary nature of basing an information system on a second-order differential equation was, of course, recognized by Corliss [5].

combination, provided $\sum_i (\Delta f.\Delta t)_i = 1/2$. In case (iii) we might have $(\Delta f.\Delta t)_c + (\Delta f.\Delta t)_u + (\Delta f.\Delta t)_v = \sum_i (\Delta f.\Delta t)_i = 1/2$, for a minimum value where $(\Delta f.\Delta t)_c = \frac{1}{10}$ and $(\Delta f.\Delta t)_u = (\Delta f.\Delta t)_v = \frac{1}{5}$; or, $(\Delta f.\Delta t)_u = \frac{1}{10}$ and $(\Delta f.\Delta t)_c = (\Delta f.\Delta t)_v = \frac{1}{5}$; or $(\Delta f.\Delta t)_v = \frac{1}{10}$ and $(\Delta f.\Delta t)_c = (\Delta f.\Delta t)_u = \frac{1}{5}$; or any other combination, provided $\sum_i (\Delta f.\Delta t)_i = 1/2$. A similar argument applies to the relation $f_0 . t_0$.

One can continue the explanation on into physiology: at the basilar membrane a traveling wave occurs with auditory stimulation which peaks at a place corresponding to the frequency of that stimulation. This peaking of the traveling wave is presumed to excite mechanically hair cell receptors. Thus, a Fourier transform is performed on the incoming signal. In cases (ii) and (iii) the center frequencies, f_0, of each $(\Delta f.\Delta t)_i$ may differ. Then different places on the basilar membrane are excited; yet when $\sum_i (\Delta f.\Delta t)_i = 1/2$, i.e. its minimum value, the information content would be similar. Even when not at its minimum value, the different dispersions of excitation would still be related to similar amounts of information.

The eigenvalues of the system in which the energy of the signal was first stored or dispersed (prior to transduction through the air) reflect more accurately the true situation at the basilar membrane. Suppose, for example, that in case (ii) $(\Delta f.\Delta t)_c + j(f_0 . t_0)_c = (\Delta f.\Delta t)_u + j(f_0 . t_0)_u = 1/2 + j1/2$ (i.e., $\sum_i (\Delta f.\Delta t)_i + j(f_0 . t_0)_i = 1 + j1$); then $\alpha_c \alpha_c^* = 1/2$ and $\alpha_u \alpha_u^* = 1/2$ and $\sum_i \alpha_i \alpha_i^* = 1$. Let us suppose, as an example, that $f_{0c} = 4$ kHz; then $t_{0c} = 125\ \mu s$, $\Delta f_c = 1000$ Hz and $\Delta t_c = 500\ \mu s$; if $f_{0u} = 8$ kHz, then $t_{0u} = 63\ \mu s$, $\Delta f_u = 2000$ Hz and $\Delta t_u = 252\ \mu s$. Compare this case with $(\Delta f.\Delta t)_c + j(f_0 . t_0)_c = \frac{3}{4} + j\frac{3}{4}$ and $(\Delta f.\Delta t)_u + j(f_0 . t_0)_u = \frac{1}{4} + j\frac{1}{4}$ [i.e., $\sum_i (\Delta f.\Delta t)_i + j(f_0 . t_0)_i = 1 + j1$ as before]. The situation is different yet the number of logons is the same. *But in the second instance* $\sum_i \alpha_i \alpha_i^* = \frac{5}{4}$, whereas this measure was 1 in the first instance. Similar arguments exist for differences between symmetric functions of differing degrees of freedom.

The Hilbert space measure, $\alpha\alpha^*$ of equation (1), is a measure of energy distribution inhomogeneity and will reference the number of degrees of freedom in the equations describing the system producing a signal as well as summed information measures. The Hilbert space measure will thus distinguish between the timbre of, say, a trumpet and a violin, as well as between a violin and a cello, because the equations describing such systems differ in the number of degrees of freedom involved. More importantly for physiologists, such a measure will correlate with events at the basilar membrane.

The general conclusion is thus: for a system of one degree of freedom there is only one phase to the Hilbert space representation; for a system of two there are two phases; for a symmetric function $y(x,y,t) = \sum c_k(t) u_k(x) v_k(y)$ or $y(x,y,t) = \overline{y(t,y,x)}$, there are three, and so on. As far as information is concerned, a machine can be defined by the way it distributes energy introduced to it.

Finally, the observation can be made that whereas Wiener's [8] signal analysis proceeded from circular functions to Fourier analysis and then to spectral analysis, due to the above considerations, it is, perhaps, a more basic sequence to commence with energy distribution considerations of the system, proceed to signal definition and thence to a Hilbert space representation. A spectral analysis in Hilbert space is readily available [9]. As signal representation in terms of equation (1) is more complete, the latter sequence seems to be more fundamental.

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